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Graphical formulary of statically determinate and indeterminate beams

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Abstract

From the intrinsic graphic resolution shown by the authors in the IXth International Congress of Graphic Engineering in Málaga in 1998, in this communication are made graphical chips for calculating beams with different types of support, whether they are statically determinate or indeterminate. It is shown the analogy that exists between the graphical approach, analytical and numerical. The procedure developed to make the graphical form can be extended beyond the field of structural analysis to other fields of technical application. It is intended that the presented graphical procedure serve to examine and analyze structural problems and that their use is appropriate for education.

1 Introduction

Graphical Statics is an intuitive drawing method and it comprises a set of simple techniques to solve problems of Structural Analysis when all the forces acting are contained in a single plane [1]. Because of its simplicity and handling, the graphic representation techniques (funicular polygon, the theorem of the three forces, Cremona diagram, Cullmann-Ritter method and others) were widely used during last two centuries [2-5]. Nowadays the technique of Graphic Statics has almost completely disappeared and been replaced by modern analytical and numerical methods. This displacement of the graphical method has been, mainly, by the inability to resolve, through graphic constructions, statically indeterminate beams. The graphical procedure has been limited to solve static problems, but it remains a useful tool to visualize, understand and verify the actions of groups of forces that occur in mechanical problems. The authors in the VIIIth International Congress of Graphic Engineering (Bilbao, June of 1997), presented the communication titled Graphic Method of statically determinate and indeterminate beams calculation [7]. The structural problem of the bar was dealt, translating it into graphic operations, its approach and analytical resolution. Some paths were presented to make possible the graphic calculation of stable beams under any action system and support condition. As a particular example, it was developed, in graphical chips for calculating, a beam subjected to compression or traction under any action system and support condition. The graphic resolution presented, translated the algebraic expressions that formulate the structural problem into graphic operations, which in turn, should be the algorithm that makes them programmable. It showed that the correlation between design and graphical path that resolves, is useful in practice, when it is able to solve general problems of technical application. The authors also in the IXth International Congress of Graphic Engineering (Málaga, June of 1998), presented the communication titled Application of intrinsic graphic resolution of beams [8], where they developed an example of a beam subjected to actions that generate flexion. In this paper it is developed a new approach to the intrinsic graphic resolution shown in the aforementioned communications and it is created a graphical form of beams, subjected to compression or traction, for different kind of support, whether statically determinate or indeterminate. It is intended that the improved graphical procedure will serve to examine and analyze structural problems, so that its transmission is ideal for education.

2 Intrinsic Method of Structural Analysis

The Intrinsic Method of Structural Analysis [9-11] is a kind of analysis that addresses the whole linear structural problem through a single system of linear differential equations (equation of the effect in the section) and it resolves through analytical, numerical and graphical procedures. The model starts defining the concepts of the linear structural problem and the principles and assumptions applied in Strength of Materials. It also expresses the formal and material conditions of the structural design, the system of action and supporting conditions, application of these principles and assumptions. The model relates the expressions above, using the usual laws of equilibrium, behavior and compatibility, trough geometry and differential vector analysis. These relationships are shown in the equation of the effect in the section (→ Appendix: Formulation), whose global expression is [10]:

$$\begin{aligned}
&DV_x &&+ q_x = 0\\
&DV_y &&+ q_y = 0\\
&DV_z &&+ q_z = 0\\
&+\upsilon_{ty}V_z + DM_x &&+ m_x = 0\\
&-\upsilon_{tx}V_z + DM_y &&+ m_y = 0\\
&-\upsilon_{ty}V_x + \upsilon_{tx}V_y + DM_z &&+ m_z = 0\\
&-\gamma_{xx}M_x - \gamma_{yx}M_y - \gamma_{zx}M_z + D\theta_x &&- \Theta_x = 0\\
&-\gamma_{xy}M_x - \gamma_{yy}M_y - \gamma_{zy}M_z + D\theta_y &&- \Theta_y = 0\\
&-\gamma_{xz}M_x - \gamma_{yz}M_y - \gamma_{zz}M_z + D\theta_z &&- \Theta_z = 0\\
&-\varepsilon_{xx}V_x - \varepsilon_{yx}V_y - \varepsilon_{zx}V_z + \upsilon_{ty}\theta_z + D\delta_x &&- \Delta_x = 0\\
&-\varepsilon_{xy}V_x - \varepsilon_{yy}V_y - \varepsilon_{zy}V_z - \upsilon_{tx}\theta_z + D\delta_y &&- \Delta_y = 0\\
&-\varepsilon_{xz}V_x - \varepsilon_{yz}V_y - \varepsilon_{zz}V_z - \upsilon_{ty}\theta_x + \upsilon_{tx}\theta_y + D\delta_z &&- \Delta_z = 0
\end{aligned} \tag{1}$$

This equation of the effect in the section (1), organized and expressed in compact form can be scored as follows:

$$\frac{d\mathbf{e}(s)}{ds} = \left[\mathbf{T}_\mathbf{D}(s)\right]\mathbf{e}(s) + \mathbf{q}(s) \tag{2}$$

The model solves this formulation by analytical, numerical and graphical procedures. Three resolutions involve similar operations. Starting with the system of the effect in the section, it is always gotten a bundle of solutions to the problem that depends on the shape, material and system of actions. The stable solution (the only one), takes shape when the support conditions are applied on the bundle of solutions, that are derived from the same mechanical design, but which are treated in the methods of resolution as the boundary conditions of the mathematical problem.

3 Relationship between intrinsic resolutions

The Intrinsic Method of Structural Analysis [9-11] is a Numeric and graphic resolutions are the reflection of the intrinsic analytical resolution in the repeated addends of the equations that are successively integrated [11]. The analytical bundle of solutions to the equation of the effect in the section (2) is:

$$\mathbf{e}(s) = \left[\mathbf{T}(s)\right]\mathbf{e}(s_\mathrm{I}) + \mathbf{Q}(s) \tag{3}$$

Being: $\left[\mathbf{T}(s)\right] = e^{\int_{s_\mathrm{I}}^{s}\left[\mathbf{T}_\mathbf{D}(s)\right]ds}$ Transfer Matrix (data).

$$\mathbf{Q}(s) = e^{\int_{s_\mathrm{I}}^{s}\left[\mathbf{T}_\mathbf{D}(s)\right]ds}\int_{s_\mathrm{I}}^{s}\mathbf{q}(s)\, e^{-\int_{s_\mathrm{I}}^{s}\left[\mathbf{T}_\mathbf{D}(s)\right]ds}\, ds$$

Integrated action vector (data).

The only solution is materialized when the support conditions are applied to the analytical bundle of solutions (3). When the effect in the section value is determinate in the initial end of the linear resistive element, $\mathbf{e}(s_\mathrm{I})$, $\mathbf{e}(s)$ effect is known in any point of the piece.

The new graphic resolution presented translates the previous analytical expressions into graphic operations, which in turn, constitute an algorithm which makes them programmable. In fig. 1 this translation is displayed and it is shown the graphic bundle of solutions.

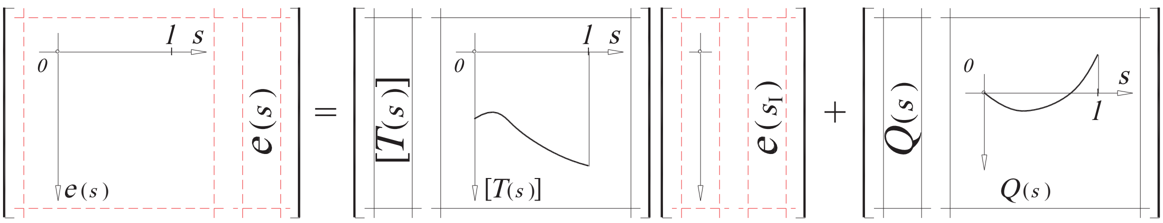

Fig. 1 Graphic bundle of solutions of the effect in the section

$\mathbf{e}(s)$, $\mathbf{e}(s_\mathrm{I})$ are unknown values, and they are represented with the picture on the left and the formula on the right in each bracket, while data $\mathbf{Q}(s)$, $\left[\mathbf{T}(s)\right]$, backwards.

Determined graphically the effect on the leading end section of the linear resistive element to apply the conditions of support, you can determine the effect anywhere in the piece. In fig. 2 it is shown the graphical solution.

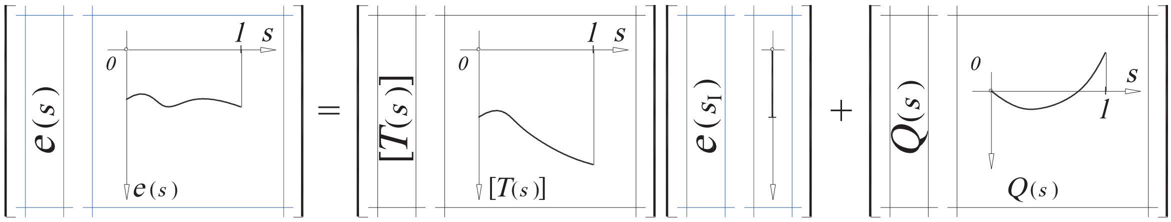

Fig. 2 Intrinsic graphical solution of the effect in the section

This representation all values are data and therefore are expressed in each bracket with the formula on the left and its right associated drawing. On the graph above can be directly measured to obtain the value of the effect in the section. It is considered feasible to use the presented graphical method to solve the structural problem of linear resistive element.

4 Beam subjected to compression or traction

In the case of the bar with overlap between the section axis and the main axis of inertia, the equation of the effect in section (1) breaks down into four sub-systems of the effect in the section [12]. Two of them are second order and the other two are fourth order.

These are the effects of traction-compression beam and the torsion beam on one side and on another side bending beam in each of the two normal axes. It is presented the graphic resolution of the effect equation of the traction-compression beam (4) because of its convenience to expose operations. Any other case is done similar.

$$\begin{aligned} \frac{\square\square}{\square x} \qquad + q_x &= 0 \\ -\frac{\square}{\square\square} + \frac{\square\square}{\square\square} - \Lambda_x &= 0 \end{aligned} \qquad (4)$$

The formulation of the system of the traction-compression beam effect is obtained from the equilibrium of forces and compatibility of displacements in the tangential axis of a guideline differential element. The fig. 3 shows graphically the differential analysis made to obtain the system.

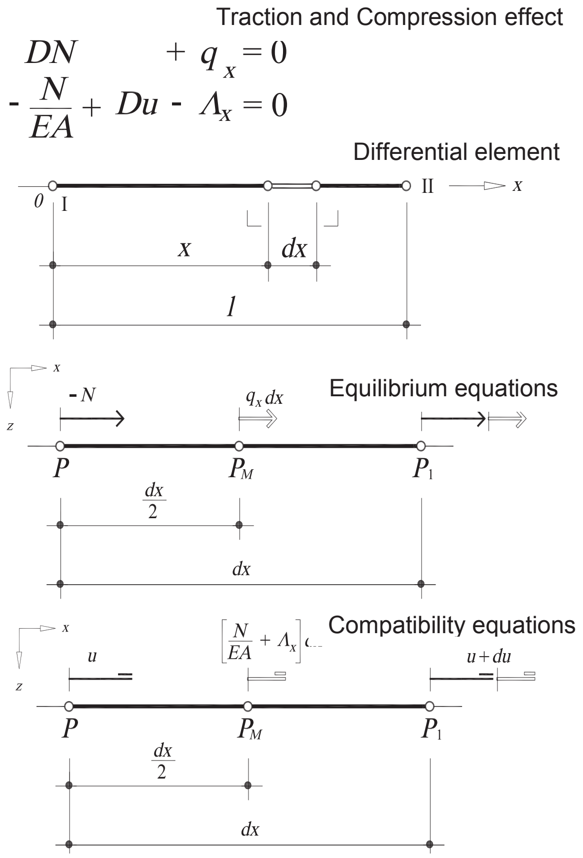

Fig. 3 Traction-compression beam effect

It is considered that this kind of graphic deduction of a differential system formulation is perfect for transmission and exposition.

Uniform Force Action

The differential system shown before (4), in the particular case of analyzing a beam with the same material and the same section along its length, under a uniform force action has the following analytical bundle of solutions:

$$\begin{aligned} \square(x) &= \square(0) \qquad - qx \\ \square(x) &= \frac{\square(0)\,x}{\square\square} + \square(0) - \frac{qx^2}{2\,\square\square} \end{aligned} \qquad (5)$$

The new bundle of solutions can be seen in the image below (see fig. 4).

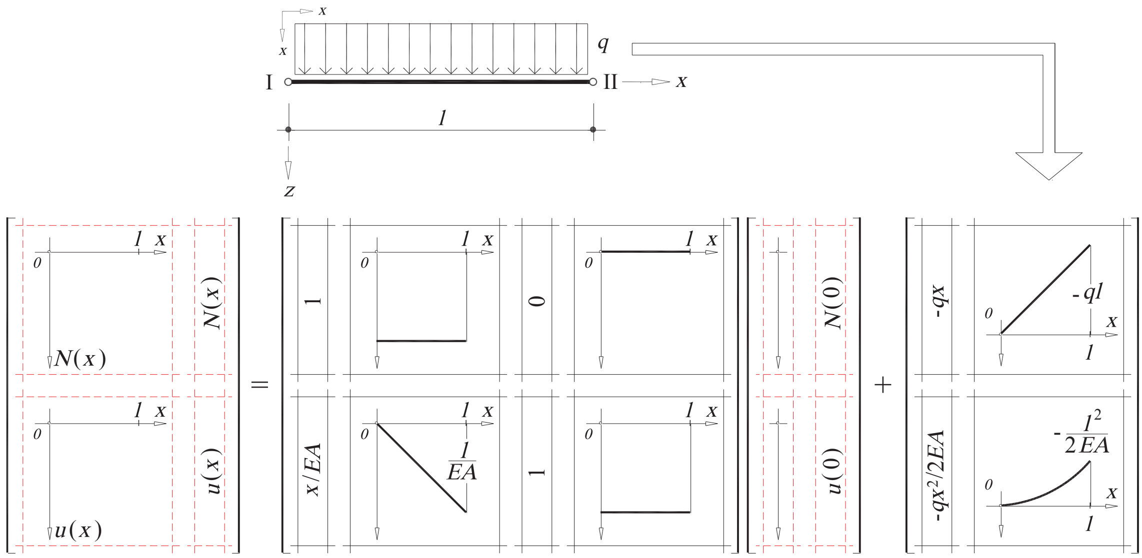

Fig. 4 Graphic bundle of solutions of the traction-compression beam under a constant force action.

For beams with the same material and section, graphic matrix does not change. The last graphic vector in the fig. 4 only depends on the action system. It is also possible to graph (see fig. 5) the relationship between the values of the effect at the end of a beam subjected to traction-compression.

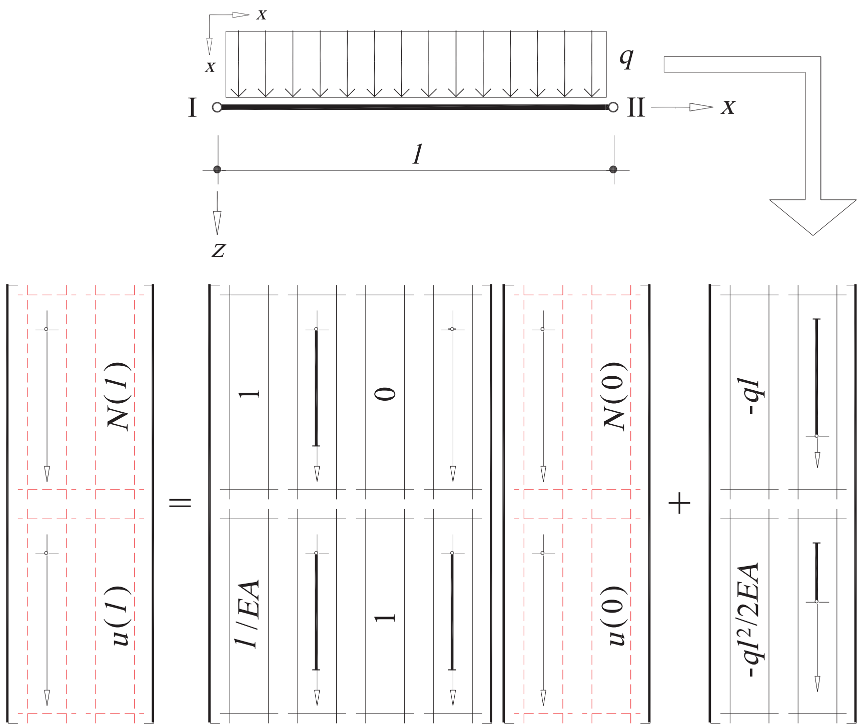

Fig. 5 Relationship between the effects at the end of a beam subjected to traction-compression under a constant force action

In this case of effect in the section, the support only reports on two null values (one for each end of the beam) of the four that are unknown. In the general case are reported twelve known values of the effect in the ends, six for each support. It is possible, by graphic operations, to solve the unknown values of the effect in the supports. For each type of support the effect values at the end will be different. Knowing the effect on the leading end of the beam its value is inserted in the bundle of solutions and the effect is determined at any point in the guideline of the beam. Fig. 6 represents the values of the effect at the ends of the traction - compression beam under a uniform force action and the intrinsic graphic solution.

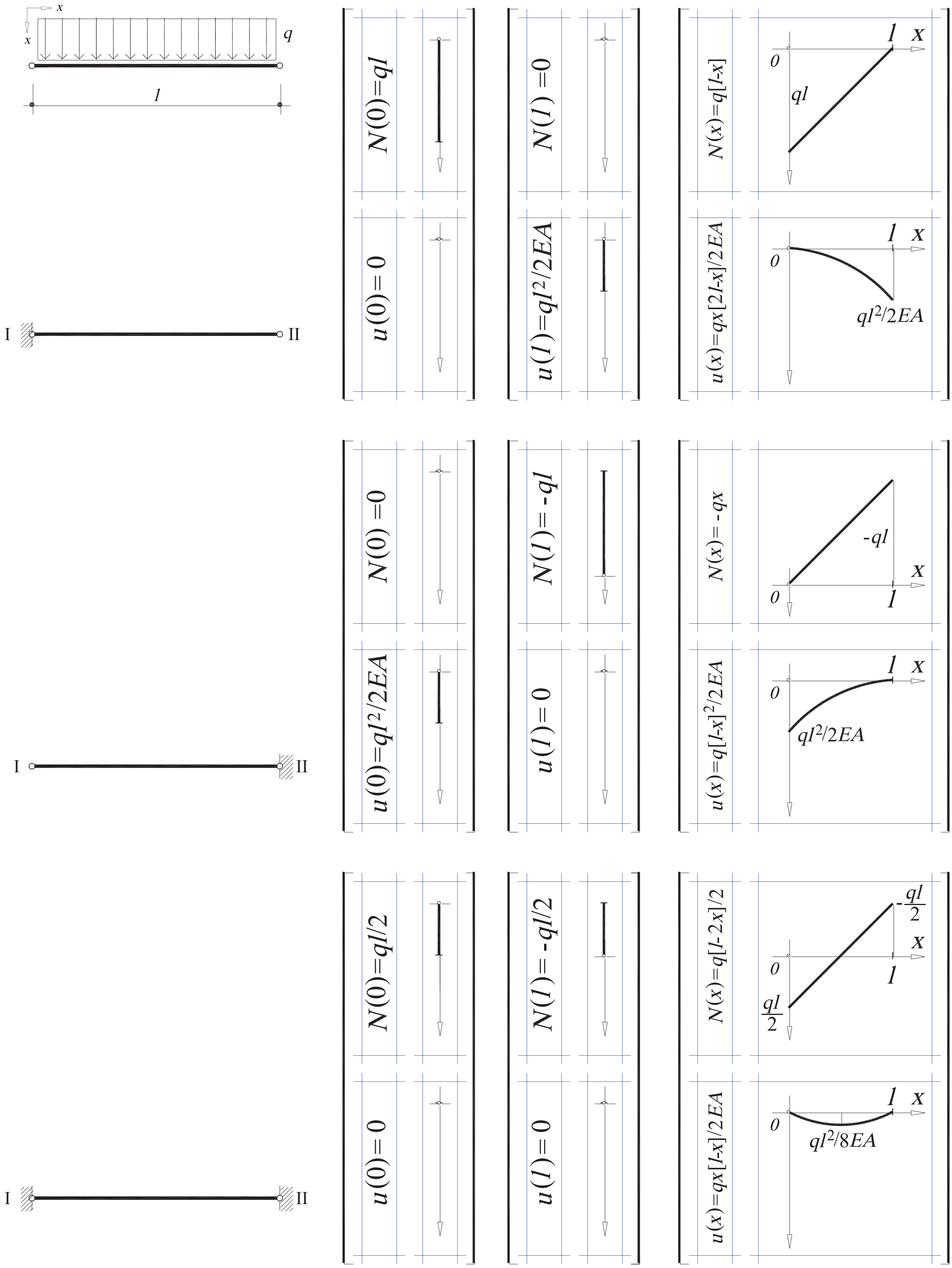

Fig. 6 Chip for calculating a traction – compression beam under a constant force action

Point Force Action

The differential system (4), in the particular case of analyzing a beam with the same material and section along its length, under a point force action in $\alpha l \left(0 \le \alpha \le 1\right)$, has the following analytical bundle of solutions:

$$N(x) = N(0) \qquad -\Psi$$
$$u(x) = \frac{N(0)\,x}{EA} + u(0) - \frac{\Psi\left[x-\alpha l\right]}{EA} \qquad (6)$$

Where: $x \le \alpha l \Rightarrow \Psi = 0 ; x \ge \alpha l \Rightarrow \Psi = Q$

Graphic bundle of solutions is shown in the next fig. 7:

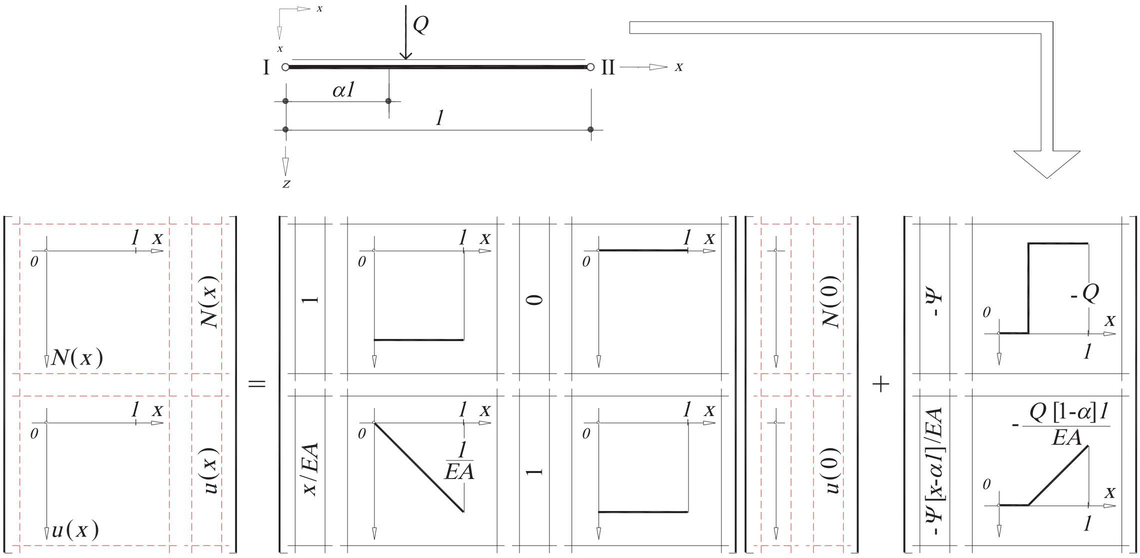

Fig. 7 Graphic bundle of solutions of a traction – compression beam under a point force action

It is graphically shown (fig. 8) the relationship between the values of the effect at the ends of a traction – compression beam.

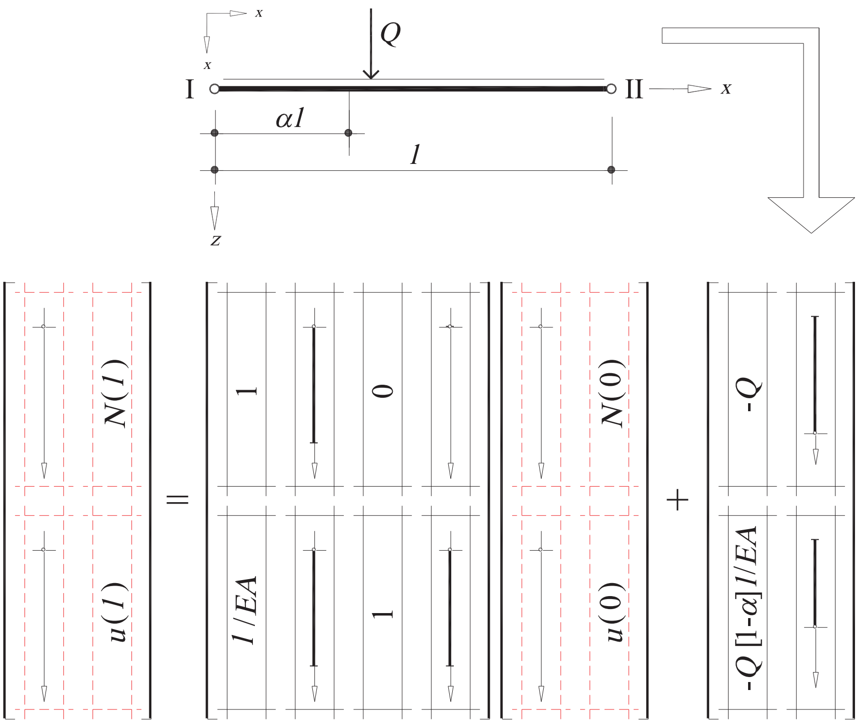

Fig. 8 Relationship between the effects at the ends of a traction – compression beam under a point force action

Knowing the effect on the leading end of the beam, to perform the necessary graphic operations, its value is entered in the bundle of solutions and the effect is determined at any point of the guideline of the beam. Applying the three possible supports casuistic, fig. 9, represents the values of the effect at the ends of the traction - compression beam under a point force action and the intrinsic graphic solution.

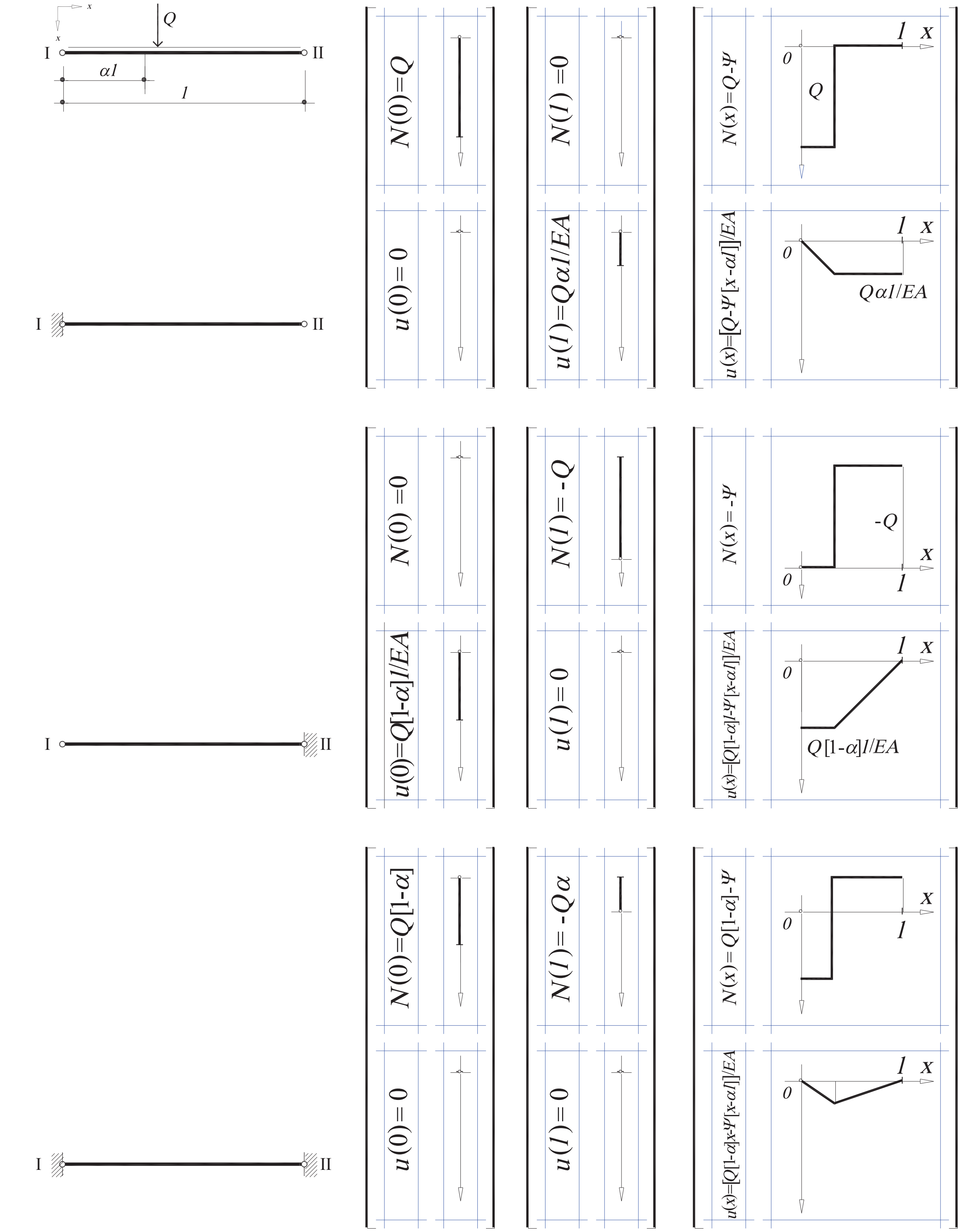

Fig. 9 Chip for calculating a traction – compression beam under a point force action

5 Conclusions

The Graphic Statics only addresses structural problems of isostatic beams. As is tried to show in this communication, it is possible to graphically address both statically determinate beams calculation and indeterminate beams calculation by the explained method, the Graphic Statics can be developed beyond its traditional boundaries.

Based on the differential analysis of a structural element, rather than static equilibrium conditions, it is possible to proceed to calculate any beam under any system of action, with all kind of support, through a known fundamental graphical construction, similar to the funicular polygon as graphical integration. For the same data of shape and material, graphic matrix does not change. In turn, for each system of action, the graphic vector (independent integrals and succesive) is the same. Making auxiliary paths of these terms and annotating them orderly in sum, the graphic bundle of solutions of each structural problem does not depend on the type of support. For the same conditions of shape, material and action, the bundle of solutions is common. The determination of the vector of the effect in the ends, varies depending on the support conditions, and it can be made by its corresponding graphic path. The intrinsic graphic solution of each structural problem is a set of final graphic multiplications and sums.

The intrinsic graphic resolution, departs from the Graphic Statics own procedures, because includes under one method, the cases of isostaticity and hyperstaticity. The structure of graphic chips for calculating shows that the resolution procedure (set of graphic operations) is unique for all kind of problems. The different starting conditions of each structural problem involve more o less graphic paths to carry out. Most of them are common in different problems. Under systematic procedure conditions (which can be automated and provided in tables) the graphical resolution of the structural problem of the beam is useful in the practice. The own knowledge of Graphic Expression in Engineering is the necessary conceptual support to address technical problems of graphic calculating.

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Appendix: Formulation

A curved beam is generated by a plane cross-section which centroid □ sweeps perpendicularly through all the points of an axis line. The vector radius $\mathbf{r} = \mathbf{r}(s)$ expresses this curved line, where s (length of the arc) is the independent variable of the structural problem. The reference coordinate system used to represent the intervening known and unknown functions of the problem is the Frenet frame $\square_{t\square\square}$. Its unit vectors tangent $\mathbf{t}$, normal $\mathbf{n}$ and binormal $\mathbf{b}$ are:

$$\mathbf{t} = D\mathbf{r}\,;\ \mathbf{n} = D^2\mathbf{r} \big/ \left|D^2\mathbf{r}\right|\,;\ \mathbf{b} = \mathbf{t} \times \mathbf{n} \qquad \text{(A.1)}$$

Where, $D = \square / \square s$ is the derivative respect the parameter s. The Frenet-Serret equations describe the movement of the frame system along the axis line. They are obtained with the versors tangent, normal and binormal derivates respect the arc length. Its matricial expression is:

$$D\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix}=\begin{bmatrix}0 & \chi(s) & 0\\ -\chi(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{bmatrix}\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix} \quad \text{(A.2)}$$

Where $\chi=\chi(s)$ and $\tau=\tau(s)$ are the flexure and torsion curvatures respectively, which represent the natural equations of the centroid line.

Assuming the habitual principles and hypotheses of the strength of materials and considering the stresses associated with the normal cross-section ($\sigma,\tau_\square,\tau_\square$), the geometric characteristics of the section are: area $\square(s)$, shearing coefficients $\alpha_\square(s)$, $\alpha_{\square\square}(s)$, $\alpha_{\square\square}(s)$, $\alpha_\square(s)$, and moments of inertia $\square_t(s)$, $\square_\square(s)$, $\square_\square(s)$, $\square_{\square\square}(s)$. Longitudinal $\square(s)$ and transversal $\square(s)$ elasticity moduli give the elastic condition of the material.

Applying the equilibrium of forces, the following equation is obtained:

$$\begin{bmatrix}D & -\chi & 0\\ \chi & D & -\tau\\ 0 & \tau & D\end{bmatrix}\begin{bmatrix}\square\\ V_\square\\ V_\square\end{bmatrix}+\begin{bmatrix}q_t\\ q_\square\\ q_\square\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \quad \text{(A.3)}$$

The vectors involved in this equilibrium are:

Internal forces

$$\mathbf{V_t}=\square\,\mathbf{t}+V_\square\,\mathbf{n}+V_\square\,\mathbf{b}=\int_\square\sigma\,\square\square\,\mathbf{t}+\int_\square\tau_\square\,\square\square\,\mathbf{n}+\int_\square\tau_\square\,\square\square\,\mathbf{b}$$

Force load $\mathbf{q_t}=q_t\,\mathbf{t}+q_\square\,\mathbf{n}+q_\square\,\mathbf{b}$

The equation of moments is obtained applying the equilibrium law as well:

$$\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}\begin{bmatrix}\square\\ V_\square\\ V_\square\end{bmatrix}+\begin{bmatrix}D & -\chi & 0\\ \chi & D & -\tau\\ 0 & \tau & D\end{bmatrix}\begin{bmatrix}\square\\ M_\square\\ M_\square\end{bmatrix}+\begin{bmatrix}m_t\\ m_\square\\ m_\square\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \quad \text{(A.4)}$$

In this case, the vectors are:

Internal moments

$$\mathbf{M_t}=\square\,\mathbf{t}+M_\square\,\mathbf{n}+M_\square\,\mathbf{b}=$$

$$=\int_\square\left(\tau_\square\square-\tau_\square\square\right)\square\square\,\mathbf{t}+\int_\square\sigma\square\square\square\,\mathbf{n}-\int_\square\sigma\square\square\square\,\mathbf{b}$$

Moment load $\mathbf{m_t}=m_t\,\mathbf{t}+m_\square\,\mathbf{n}+m_\square\,\mathbf{b}$

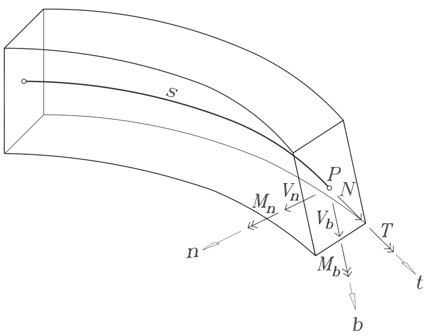

Fig. A.1 Internal forces and moments in Frenet frame.

Once the constitutive relations are defined, kinematics law relates the rotations and displacements (A.5):

$$\begin{bmatrix}-\frac{1}{\square\square_t} & 0 & 0\\ 0 & -\frac{\square}{\square(\square\square-\square^2_{\square\square})} & -\frac{\square_{\square\square}}{\square(\square\square-\square^2_{\square\square})}\\ 0 & -\frac{\square_{\square\square}}{\square(\square\square-\square^2_{\square\square})} & -\frac{\square}{\square(\square\square-\square^2_{\square\square})}\end{bmatrix}\begin{bmatrix}\square\\ M_\square\\ M_\square\end{bmatrix}+$$

$$+\begin{bmatrix}D & -\chi & 0\\ \chi & D & -\tau\\ 0 & \tau & D\end{bmatrix}\begin{bmatrix}\theta_t\\ \theta_\square\\ \theta_\square\end{bmatrix}-\begin{bmatrix}\Theta_t\\ \Theta_\square\\ \Theta_\square\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

The rotations are given by the vector $\boldsymbol{\theta}_\mathbf{t}=\theta_t\,\mathbf{t}+\theta_\square\,\mathbf{n}+\theta_\square\,\mathbf{b}$ and rotation load $\boldsymbol{\Theta}_\mathbf{t}=\Theta_t\,\mathbf{t}+\Theta_\square\,\mathbf{n}+\Theta_\square\,\mathbf{b}$.

Following the same procedure, the displacement equation is expressed (A.6):

$$\begin{bmatrix}-\frac{1}{\square\square} & 0 & 0\\ 0 & -\frac{\alpha_\square}{\square\square} & -\frac{\alpha_{\square\square}}{\square\square}\\ 0 & -\frac{\alpha_{\square\square}}{\square\square} & -\frac{\alpha_\square}{\square\square}\end{bmatrix}\begin{bmatrix}\square\\ V_\square\\ V_\square\end{bmatrix}+\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}\begin{bmatrix}\theta_t\\ \theta_\square\\ \theta_\square\end{bmatrix}+$$

$$+\begin{bmatrix}D & -\chi & 0\\ \chi & D & -\tau\\ 0 & \tau & D\end{bmatrix}\begin{bmatrix}\square\\ \square\\ \square\end{bmatrix}-\begin{bmatrix}\Delta_t\\ \Delta_\square\\ \Delta_\square\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Where displacement components are denoted as $\boldsymbol{\delta}_\mathbf{t}=\square\mathbf{t}+\square\mathbf{n}+\square\mathbf{b}$ and displacement load $\boldsymbol{\Delta}_\mathbf{t}=\Delta_t\,\mathbf{t}+\Delta_\square\,\mathbf{n}+\Delta_\square\,\mathbf{b}$.

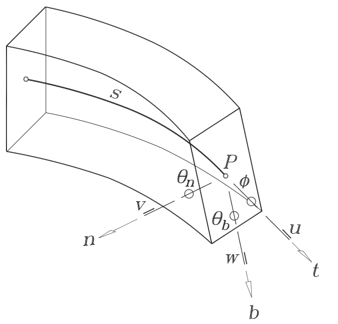

Fig. A.2 Deflections in Frenet frame.

Equations (A.3), (A.4), (A.5) and (A.6) are related and they compose the unique system of linear ordinary differential equations which simulates the structural behaviour of a curved beam element.

$$
\begin{aligned}
&D\square - \chi V_{\square} && + q_t = 0\\
&\chi\square + DV_{\square} - \tau V_{\square} && + q_{\square} = 0\\
&\tau V_{\square} + DV_{\square} && + q_{\square} = 0\\
&D\square - \chi M_{\square} && + m_t = 0\\
&- V_{\square} + \chi\square + DM_{\square} - \tau M_{\square} && + m_{\square} = 0\\
&V_{\square} + \tau M_{\square} + DM_{\square} && + m_{\square} = 0\\
&-\frac{\square}{\square\square_t} + D\theta_t - \chi\theta_{\square} && - \Theta_t = 0\\
&-\frac{M_{\square}\square}{\square\left[\square\square - \square^2_{\square\square}\right]} - \frac{M_{\square}\square_{\square\square}}{\square\left[\square\square - \square^2_{\square\square}\right]} + \chi\theta_t + D\theta_{\square} - \tau\theta_{\square} && - \Theta_{\square} = 0\\
&-\frac{M_{\square}\square_{\square\square}}{\square\left[\square\square - \square^2_{\square\square}\right]} - \frac{M_{\square}\square}{\square\left[\square\square - \square^2_{\square\square}\right]} + \tau\theta_{\square} + D\theta_{\square} && - \Theta_{\square} = 0\\
&-\frac{\square}{\square\square} + D\square - \chi\square && - \Delta_t = 0\\
&-\frac{\alpha_{\square}V_{\square}}{\square\square} - \frac{\alpha_{\square\square}V_{\square}}{\square\square} - \theta_{\square} + \chi\square + D\square - \tau\square && - \Delta_{\square} = 0\\
&-\frac{\alpha_{\square\square}V_{\square}}{\square\square} - \frac{\alpha_{\square}V_{\square}}{\square\square} + \theta_{\square} + \tau\square + D\square && - \Delta_{\square} = 0
\end{aligned}
\qquad (A.7)
$$

The system employed in the equation (A.7) is the associated to the Frenet frame in natural coordinates of the curved line. It is possible to implement a change of basis and express the functions in a global coordinate system $\square_{xyz}$ which unit vectors are $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$:

$$
\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix} = \begin{bmatrix}\upsilon_{tx} & \upsilon_{ty} & \upsilon_{tz}\\ \upsilon_{\square x} & \upsilon_{\square y} & \upsilon_{\square z}\\ \upsilon_{\square x} & \upsilon_{\square y} & \upsilon_{\square z}\end{bmatrix}\begin{bmatrix}\mathbf{i}\\ \mathbf{j}\\ \mathbf{k}\end{bmatrix} \qquad (A.8)
$$

The different coefficients of the basis change matrix represent the direction cosines between the versors of both reference coordinate systems, natural and global.
The differential system (A.7) is transformed into global Cartesian coordinates (1) where:

The components of internal forces, moments, rotations and displacements involved in the equation (1) are referred to the global absolute coordinate system.
This new general expression of the differential system, which simulates the structural comportment of the linear element, has a lower-triangular form. That important property permits to analytically solve the differential equation system using successive integrations.

$$\varepsilon_{xx} = \frac{\upsilon_{tx}^2}{\square\square} + \frac{\upsilon_{\square x}^2\alpha_{\square} + 2\upsilon_{\square x}\upsilon_{\square x}\alpha_{\square\square} + \upsilon_{\square x}^2\alpha_{\square}}{\square\square}$$

$$\varepsilon_{yy} = \frac{\upsilon_{ty}^2}{\square\square} + \frac{\upsilon_{\square y}^2\alpha_{\square} + 2\upsilon_{\square y}\upsilon_{\square y}\alpha_{\square\square} + \upsilon_{\square y}^2\alpha_{\square}}{\square\square}$$

$$\varepsilon_{zz} = \frac{\upsilon_{tz}^2}{\square\square} + \frac{\upsilon_{\square z}^2\alpha_{\square} + 2\upsilon_{\square z}\upsilon_{\square z}\alpha_{\square\square} + \upsilon_{\square z}^2\alpha_{\square}}{\square\square}$$

$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{\upsilon_{tx}\upsilon_{ty}}{\square\square} + \frac{\upsilon_{\square x}\upsilon_{\square y}\alpha_{\square} + (\upsilon_{\square x}\upsilon_{\square y} + \upsilon_{\square x}\upsilon_{\square y})\alpha_{\square\square} + \upsilon_{\square x}\upsilon_{\square y}\alpha_{\square}}{\square\square}$$

$$\varepsilon_{xz} = \varepsilon_{zx} = \frac{\upsilon_{tx}\upsilon_{tz}}{\square\square} + \frac{\upsilon_{\square x}\upsilon_{\square z}\alpha_{\square} + (\upsilon_{\square x}\upsilon_{\square z} + \upsilon_{\square x}\upsilon_{\square z})\alpha_{\square\square} + \upsilon_{\square x}\upsilon_{\square z}\alpha_{\square}}{\square\square}$$

$$\varepsilon_{yz} = \varepsilon_{zy} = \frac{\upsilon_{ty}\upsilon_{tz}}{\square\square} + \frac{\upsilon_{\square y}\upsilon_{\square z}\alpha_{\square} + (\upsilon_{\square y}\upsilon_{\square z} + \upsilon_{\square y}\upsilon_{\square z})\alpha_{\square\square} + \upsilon_{\square y}\upsilon_{\square z}\alpha_{\square}}{\square\square}$$

$$\gamma_{xx} = \frac{\upsilon_{tx}^2}{\square\square_t} + \frac{\upsilon_{\square x}^2\square + 2\upsilon_{\square x}\upsilon_{\square x}\square_{\square\square} + \upsilon_{\square x}^2\square}{\square(\square\square - \square^2_{\square\square})}$$

$$\gamma_{yy} = \frac{\upsilon_{ty}^2}{\square\square_t} + \frac{\upsilon_{\square y}^2\square + 2\upsilon_{\square y}\upsilon_{\square y}\square_{\square\square} + \upsilon_{\square y}^2\square}{\square(\square\square - \square^2_{\square\square})}$$

$$\gamma_{zz} = \frac{\upsilon_{tz}^2}{\square\square_t} + \frac{\upsilon_{\square z}^2\square + 2\upsilon_{\square z}\upsilon_{\square z}\square_{\square\square} + \upsilon_{\square z}^2\square}{\square(\square\square - \square^2_{\square\square})}$$

$$\gamma_{xy} = \gamma_{yx} = \frac{\upsilon_{tx}\upsilon_{ty}}{\square\square_t} + \frac{\upsilon_{\square x}\upsilon_{\square y}\square + (\upsilon_{\square x}\upsilon_{\square y} + \upsilon_{\square x}\upsilon_{\square y})\square_{\square\square} + \upsilon_{\square x}\upsilon_{\square y}\square}{\square(\square\square - \square^2_{\square\square})}$$

$$\gamma_{xz} = \gamma_{zx} = \frac{\upsilon_{tx}\upsilon_{tz}}{\square\square_t} + \frac{\upsilon_{\square x}\upsilon_{\square z}\square + (\upsilon_{\square x}\upsilon_{\square z} + \upsilon_{\square x}\upsilon_{\square z})\square_{\square\square} + \upsilon_{\square x}\upsilon_{\square z}\square}{\square(\square\square - \square^2_{\square\square})}$$

$$\gamma_{yz} = \gamma_{zy} = \frac{\upsilon_{ty}\upsilon_{tz}}{\square\square_t} + \frac{\upsilon_{\square y}\upsilon_{\square z}\square + (\upsilon_{\square y}\upsilon_{\square z} + \upsilon_{\square y}\upsilon_{\square z})\square_{\square\square} + \upsilon_{\square y}\upsilon_{\square z}\square}{\square(\square\square - \square^2_{\square\square})}$$